Tensorial Neural Networks: Generalization of Neural Networks and Applications to Model Compression

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Joint work with Jiahao Su, Jingling Li and Bobby Bhattacharjee

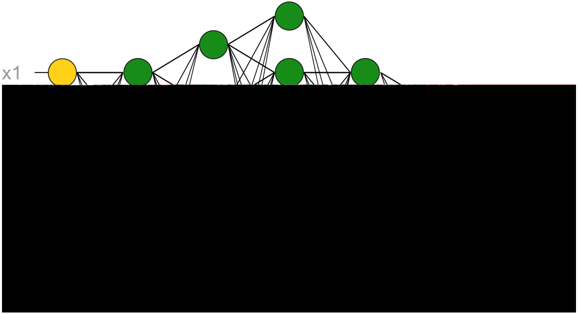
Neural Network - Nonlinear Function Approximation

Image classification

Speech recognition

Text processing

Success of Deep Neural Networks



- computation power growth
- enormous labeled data

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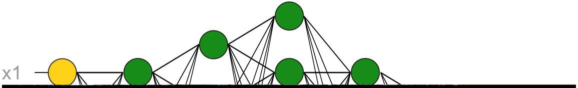


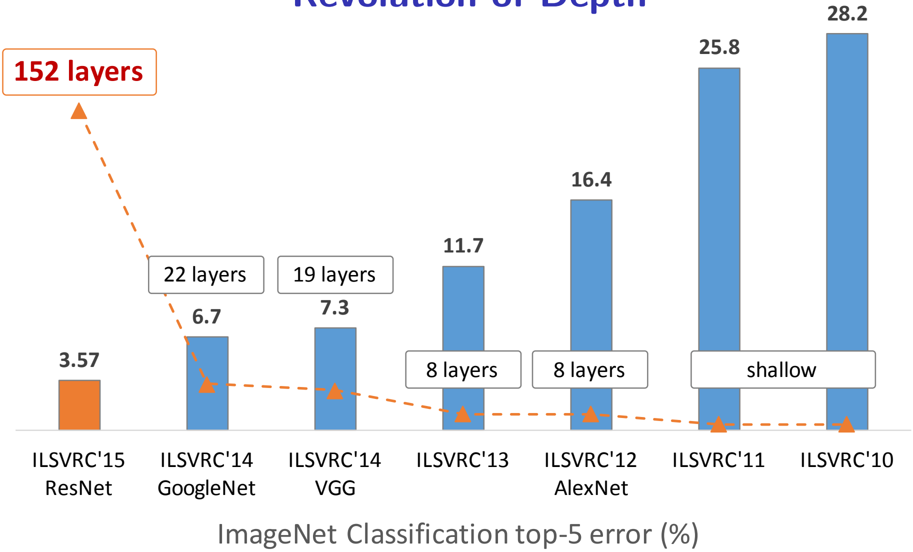
[redacted]

- computation power growth
- enormous labeled data

Express Power

- linear composition vs nonlinear composition
- shallow network vs deep structure

Revolution of Depth



Kaiming He, Xiangyu Zhang, Shaoqing Ren, & Jian Sun. “Deep Residual Learning for Image Recognition”. CVPR 2016.

- Deeper and wider architectures
- Large number of model parameters

Challenges For Large Deep Neural Network

Learning

- Learning takes longer, might not converge, susceptible to vanishing/exploding gradients, etc
- One-time cost.

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Popular Approaches

- Compress well-trained neural networks
- Find better neural network architectures

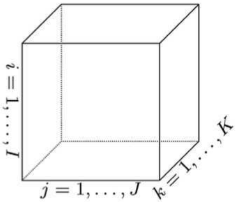
Tensorial Neural Networks (TNNs)

- Achieve both compression and better architecture
- Generalize $\left\{ \begin{array}{l} \text{matrix-vector product} \\ \text{convolution} \end{array} \right.$ to **general tensor operations**
- New tensor algebra: extend existing operations with low order operands to those with high order operands

What is a tensor? How to denote tensors effectively?

Multi-dimensional Array

- Tensor - Higher order matrix
- The number of dimensions is called tensor order.



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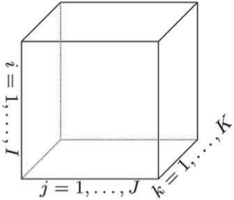
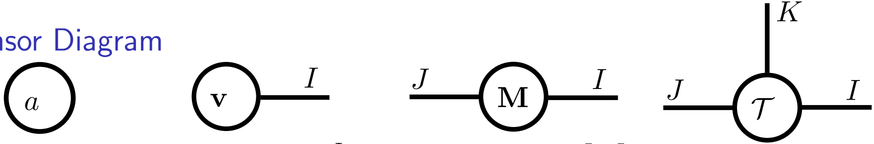
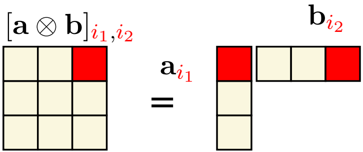


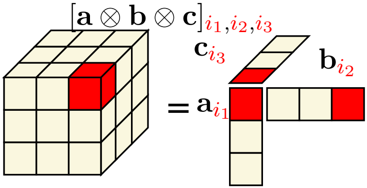
Tensor Diagram



Scalar $a \in \mathbb{R}$ | Vector $\mathbf{v} \in \mathbb{R}^I$ | Matrix $\mathbf{M} \in \mathbb{R}^{I \times J}$ | Tensor $\mathcal{T} \in \mathbb{R}^{I \times J \times K}$

Tensor Product





- $[\mathbf{a} \otimes \mathbf{b}]_{i_1,i_2} = \mathbf{a}_{i_1} \mathbf{b}_{i_2}$
- Rank-1 matrix

- $[\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}]_{i_1,i_2,i_3} = \mathbf{a}_{i_1} \mathbf{b}_{i_2} \mathbf{c}_{i_3}$
- Rank-1 tensor

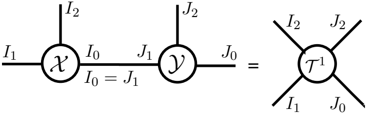
- Existing tensor operations are only defined on lower-order $\mathcal{X}$ and $\mathcal{Y}$ such as matrices and vectors.

Generalized Tensor Operations

Generalized tensor operations on *High-order tensor operands*

Mode-(0,1) tensor contraction

$\mathcal{X} \times_1^0 \mathcal{Y} \rightarrow \mathcal{T}^1$



$$\mathcal{T}^1_{i_1,i_2,j_0,j_2} = \sum_r \mathcal{X}_{r,i_1,i_2}\mathcal{Y}_{j_0,r,j_2}$$

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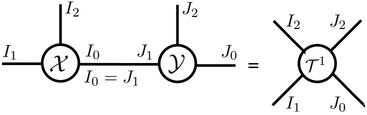

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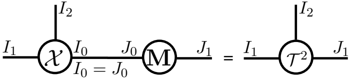
Mode-0 tensor product

$\mathcal{X} \times_0 \mathbf{M} \to \mathcal{T}^2$



$\mathcal{T}^2_{j_0,i_1,i_2} = \sum_r \mathcal{X}_{r,i_1,i_2} \mathbf{M}_{j_0,r}$

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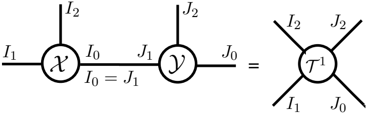

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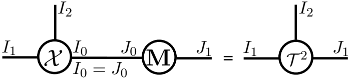
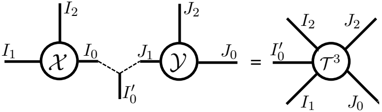


$\mathcal{T}^2_{j_0,i_1,i_2} = \sum_r \mathcal{X}_{r,i_1,i_2}\mathbf{M}_{j_0,r}$

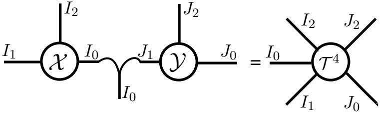
Mode-(0,1) tensor convolution

$\mathcal{X} *_1^0 \mathcal{Y} \to \mathcal{T}^3$



$\mathcal{T}^3_{:,i_1,i_2,j_0,j_2} = \mathcal{X}_{:,i_1,i_2} * \mathcal{Y}_{j_0,:,j_2}$

Mode-(0,1) tensor partial outer product $\mathcal{X} \otimes_1^0 \mathcal{Y} \to \mathcal{T}^4$

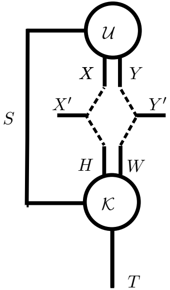


$\mathcal{T}^4_{r,i_1,i_2,j_0,j_2} = \mathcal{X}_{r,i_1,i_2}\mathcal{Y}_{j_0,r,j_2}$

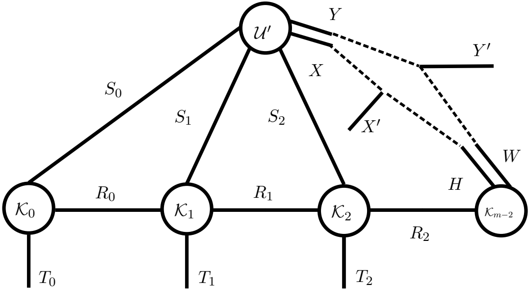
- Similar definitions apply to general mode-(i,j) tensor operations.

Outline

One-layer of CNN vs One-layer of TNN

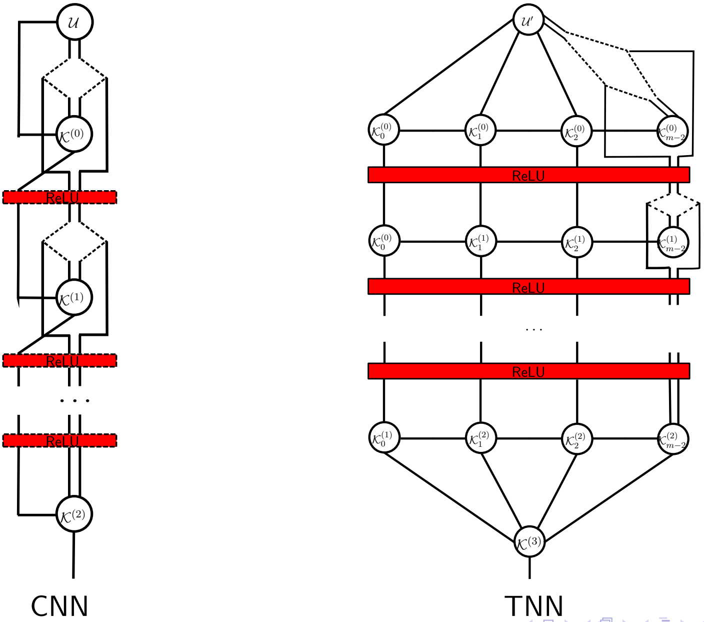


One layer of CNN



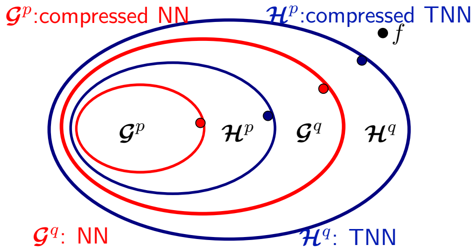
One layer of TNN

CNN vs TNN


$\mathcal{U}$
$\mathcal{K}^{(0)}$
ReLU
$\mathcal{K}^{(1)}$
ReLU
...
ReLU
$\mathcal{K}^{(2)}$
CNN
$\mathcal{U}'$
$\mathcal{K}_0^{(0)}$ $\mathcal{K}_1^{(0)}$ $\mathcal{K}_2^{(0)}$ $\mathcal{K}_{m-2}^{(0)}$
ReLU
$\mathcal{K}_0^{(0)}$ $\mathcal{K}_1^{(1)}$ $\mathcal{K}_2^{(1)}$ $\mathcal{K}_{m-2}^{(1)}$
ReLU
...
ReLU
$\mathcal{K}_0^{(1)}$ $\mathcal{K}_1^{(2)}$ $\mathcal{K}_2^{(2)}$ $\mathcal{K}_{m-2}^{(2)}$
$\mathcal{K}^{(3)}$
TNN

Tensorial Neural Networks

Relationship between NNs and TNNs



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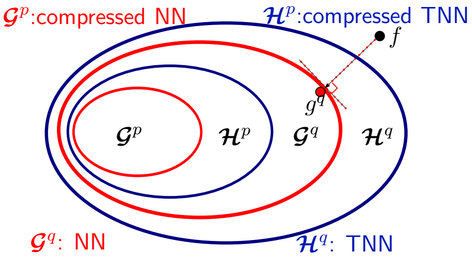

- **Learning** of a NN with q parameters: g^q, closest to f in $\mathcal{G}^q$

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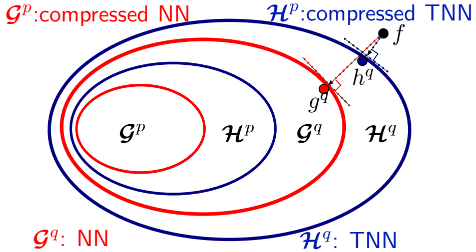

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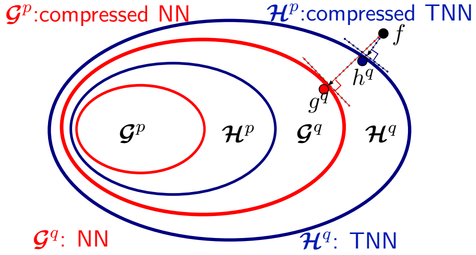

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$$\boxed{h^q \textbf{ is closer to } f \textbf{ than } g^q}$$

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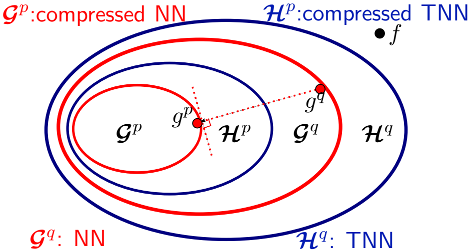

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- **Compression** of g^q to p parameters: g^p, closest to g^q in $\mathcal{G}^p$

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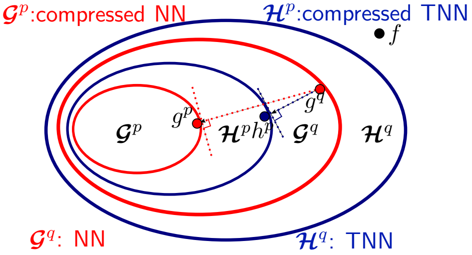

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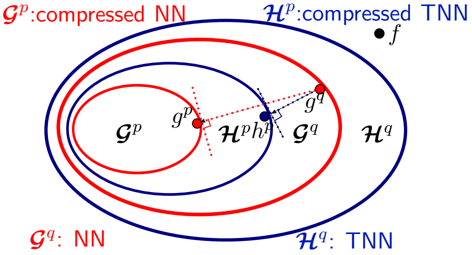

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- **Compression** of g^q to p parameters: h^p, closest to g^q in $\mathcal{H}^p$

$\boxed{\textbf{Compressed TNN } h^p \textbf{ is closer to pre-trained } g^q \textbf{ than } g^p}$

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Compression Using Invariant Structure In Deep Neural Networks

Common Compression Techniques

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Low Rank Approximation

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- Reduce the number of parameters by a factor **polynomial** in the dimension
  - Caveat: only when the weight matrices (convolutional kernels) are **low rank**

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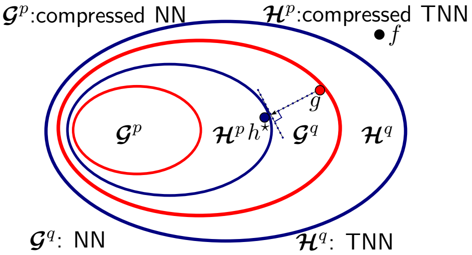
Exploiting other invariant structure via low rank approximation?

Periodicity, modulation and low rank?

Idea for Compression using TNN

Given a pre-trained NN $g \in \mathcal{G}^q$, how do we find a TNN $h^\star \in \mathcal{H}^p$ that is as close to g as possible?



1. Tensorization
2. Generalized Tensor Decomposition
3. Mapping NN to TNN
   - End-to-End (E2E) Learning: traditional learning approach
   - Sequential (Seq) Learning: learning each layer from bottom-up sequentially

Tensorization

Reshape the object to a higher order object

- Identifying periodic and modulated structure by exploiting the low rank structure in the reshaped matrix

Toy Example

Invariant structures

$$\left\{\begin{array}{c} [1, 2, 3, 1, 2, 3, 1, 2, 3] \\ \text{Periodic structure} \\ \\ [1, 1, 1, 2, 2, 2, 3, 3, 3] \\ \text{Modulated structure} \end{array}\right. \xrightarrow{\text{reshape}} \begin{bmatrix} 1, 1, 1 \\ 2, 2, 2 \\ 3, 3, 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1, 1, 1 \end{bmatrix}$$

Low-rank structure

Component

Tensorization

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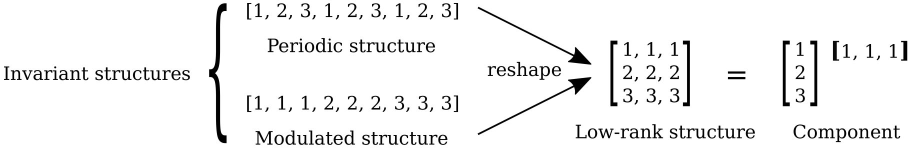
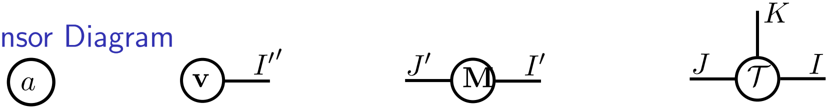
Toy Example



Tensor Diagram



Scalar $a \in \mathbb{R}$ Vector $\mathbf{v} \in \mathbb{R}^{I''}$ Matrix $\mathbf{M} \in \mathbb{R}^{I' \times J'}$ Tensor $\mathcal{T} \in \mathbb{R}^{I \times J \times K}$

- Reshape $\mathbf{v}$ to $\mathbf{M}$ to $\mathcal{T}$
  - $I'' = I' \times J' = I \times J \times K$

Higher Order Tensor Decompositions

$\boxed{m\text{-order tensor } \mathcal{T} \in \mathbb{R}^{I_0 \times I_1 \times \cdots \times I_{m-1}}}$

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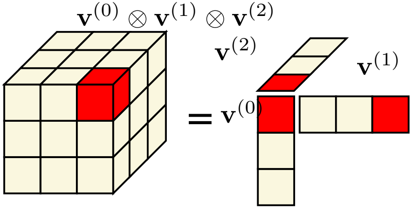
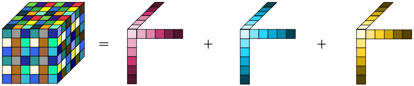
CANDECOMP/PARAFAC (CP) Decomposition

- Factorize a tensor into sum of rank-1 tensors
- Rank-1 tensor is defined as outer product of multiple vectors
- $\mathcal{T} = \sum_{r=0}^{R-1} \mathbf{v}_r^{(0)} \otimes \cdots \otimes \mathbf{v}_r^{(m-1)}$

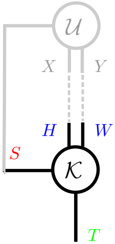




- $[\mathbf{v}^{(0)} \otimes \mathbf{v}^{(1)} \otimes \mathbf{v}^{(2)}]_{i_1, i_2, i_3} = \mathbf{v}_{i_1}^{(0)} \mathbf{v}_{i_2}^{(1)} \mathbf{v}_{i_3}^{(2)}$
- Rank-1 tensor

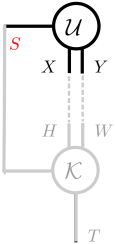
- $\sum_{r=0}^{2} \mathbf{v}_r^{(0)} \mathbf{v}_r^{(1)} \mathbf{v}_r^{(2)}$
- Rank-3 tensor

- **Convolutional Kernel**: $\mathcal{K} \in \mathbb{R}^{H \times W \times S \times T}$



Compression of Convolutional Layer w/ Tensor Decompositions

- **Convolutional Kernel**: $\mathcal{K} \in \mathbb{R}^{H \times W \times S \times T}$
- **Input** tensor: $\mathcal{U} \in \mathbb{R}^{X \times Y \times S}$



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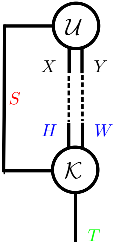
- Map the input tensor $\mathcal{U}$ to an **output** tensor $\mathcal{V} \in \mathbb{R}^{X' \times Y' \times T}$:

$$\mathcal{V}_{x,y,t} = \sum_{s=0}^{S-1} \sum_{i,j} \mathcal{K}_{i,j,s,t}\, \mathcal{U}_{x-i,y-j,s}.$$



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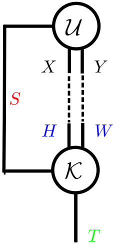
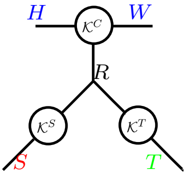


CP Decomposition on the Kernel

- **CP**: Decompose kernel $\mathcal{K}$ into 3 factor tensors
- $\mathcal{K}_{i,j,s,t} = \sum_{r=0}^{R-1} \mathcal{K}^S_{s,r} \, \mathcal{K}^C_{i,j,r} \, \mathcal{K}^T_{r,t}$
- # of param.: $HWST \to (HW + S + T)R$



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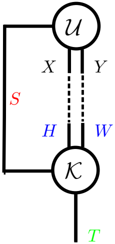
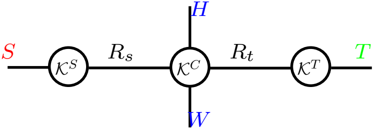


TK Decomposition on the Kernel

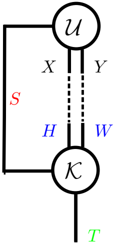
- **TK**: Decompose $\mathcal{K}$ into 1 core tensor, 2 factor tensors
- $\mathcal{K}_{i,j,s,t} = \sum_{r_s=0}^{R_s-1} \sum_{r_t=0}^{R_t-1} \mathcal{K}^S_{s,r_s}\ \mathcal{K}^C_{i,j,r_s,r_t}\ \mathcal{K}^T_{r_t,t}$
- # of param.: $HWST \to SR_s + HWR_sR_t + R_tT$



Compression of Convolutional Layer w/ Tensor Decompositions

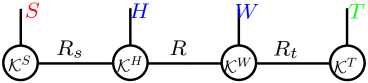
- **Convolutional Kernel**: $\mathcal{K} \in \mathbb{R}^{H\times W\times S\times T}$
- **Input** tensor: $\mathcal{U} \in \mathbb{R}^{X\times Y\times S}$
- Map the input tensor $\mathcal{U}$ to an **output** tensor $\mathcal{V} \in \mathbb{R}^{X'\times Y'\times T}$:

$$\mathcal{V}_{x,y,t} = \sum_{s=0}^{S-1} \sum_{i,j} \mathcal{K}_{i,j,s,t}\, \mathcal{U}_{x-i,y-j,s}.$$



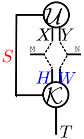
TT Decomposition on the Kernel

- **TT**: Decompose $\mathcal{K}$ into 4 factor tensors

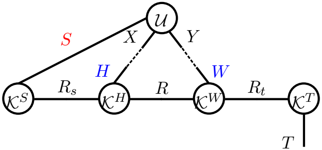


- $\mathcal{K}_{i,j,s,t} = \sum_{r_s=0}^{R_s-1} \sum_{r=0}^{R-1} \sum_{r_t=0}^{R_t-1} \mathcal{K}^S_{s,r_s} \mathcal{K}^H_{r_s,i,r}\, \mathcal{K}^W_{r,j,r_t}\, \mathcal{K}^T_{r_t,t}$
- # of param.: $HWST \rightarrow SR_s + HR_sR + WR_tR + R_tT$

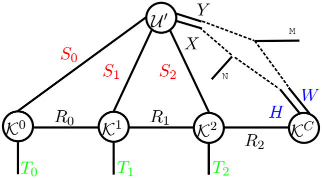
Reshaped Tensor Decomposition— Narrower & Deeper Nets



Uncompressed



Compressed via TT



Compressed via r-TT $(m = 3)$

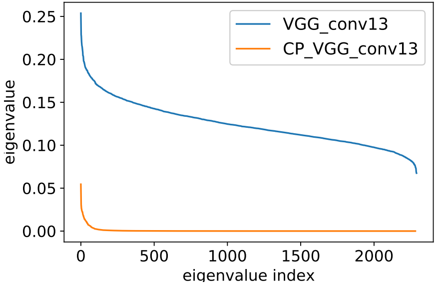
TT Decomposition on the Reshaped Kernel

- Param. $\#: HWST \rightarrow SR_s + HR_sR + WR_tR + R_tT \rightarrow (m(ST)^{\frac{1}{m}}R + HW)R$

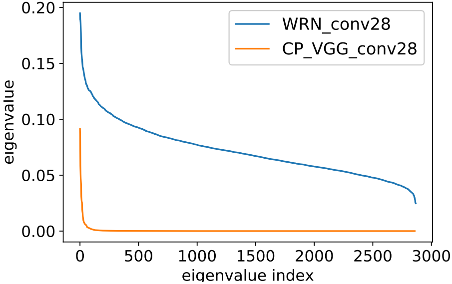
Low Rank Structure?

Comparisons of Eigenvalue Spectra

- CP-VGG and CP-WRN are TNNs



VGG (layer 13)



WRN (layer 28)

Outline

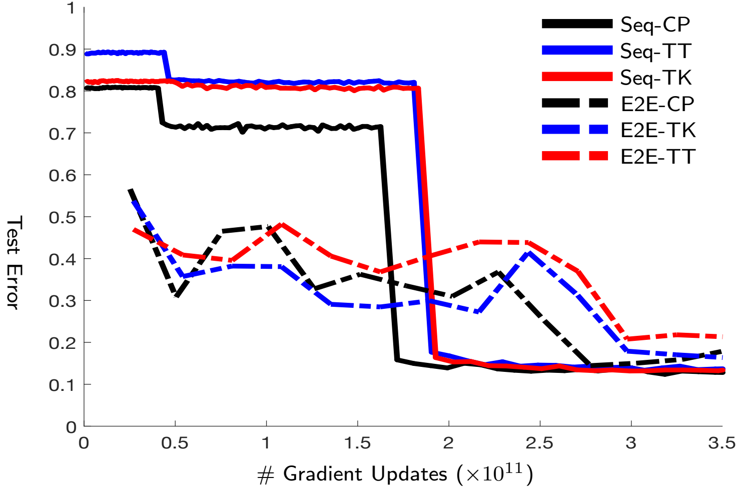
Experiments - Compress CIFAR10 Resnet-32

Successful Compression of CIFAR10 Resnet-32 Network (Su, Li, Bhattacharjee & H., 2018)

	Compression rate					Compression rate			
	5%	10%	20%	40%		2%	5%	10%	20%
SVD	83.09	87.27	89.58	90.85	r-TR†	-	80.80	-	90.60
CP	84.02	86.93	88.75	88.75	r-CP	85.7	89.86	**91.28**	-
TK	83.57	86.00	88.03	89.35	r-TK	61.06	71.34	81.59	87.11
TT	77.44	82.92	84.13	86.64	r-TT	78.95	84.26	87.89	-

- Testing accuracies of tensor methods under compression rates.
- The uncompressed network achieves 93.2% accuracy.
- CIFAR10 Resnet-32 has 0.46M parameters that have to be trained and retained during testing.

Experiments - Convergence Rate



Convergence rate for Seq vs. E2E compression on CIFAR10.

Experiments - Compress ImageNet Resnet-50

Successful Compression of ImageNet Resnet-50 Network (Su, Li, Bhattacharjee & H., 2018)

# samples	Uncompressed # params.: 25M	TT (E2E) # params.: 2.5M	r-TT (Seq) # params.: 2.5M
0.24M	4.22	2.78	44.35
0.36M	6.23	3.99	46.98
0.60M	9.01	7.48	49.92
1.20M	17.3	12.80	52.59
2.40M	30.8	18.17	**54.00**

- Testing accuracy of tensor methods compared to the uncompressed ImageNet Resnet-50.

Summary

1. Extends traditional NN via a new framework TNN which
   - naturally preserve multi-dimensional structures of the input data (such as videos)
   - effectively compress existing NN by exploiting additional invariant structures
2. Introduce a system of *generalized tensor algebra* and *generalized tensor operations* for
   - efficient learning and prediction in TNNs
   - deriving and analyze backpropagation rules for generalized tensor operations.
3. Interpretations of famous neural network architectures using TNNs.

Arxiv: 1805.10352

Code: github.com/FurongHuang/TTP-NeuralNets-Compression